ON INHOMOGENEOUS p-ADIC POTTS MODEL ON A CAYLEY TREE

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Abstract

We consider a nearest-neighbor inhomogeneous p-adic Potts (with $q \geq 2$ spin values) model on the Cayley tree of order $k \geq 1$. The inhomogeneity means that the interaction J_{xy} couplings depend on nearest-neighbors points x, y of the Cayley tree. We study ($p-$ adic) Gibbs measures of the model. We show that (i) if $q \notin p\mathbb{N}$ then there is unique Gibbs measure for any $k \geq 1$ and $\forall J_{xy}$ with $|J_{xy}| < p^{-1/(p-1)}$. (ii) For $q \in p\mathbb{N}$, $p \geq 3$ one can choose J_{xy} and $k \geq 1$ such that there exist at least two Gibbs measures which are translation-invariant.

Keywords: p-adic field, Potts model, Cayley tree, Gibbs measure

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1 Introduction

In the paper we consider models with a nearest neighbor interactions in the field of $p-$adic numbers on Cayley tree. The classical (real value) example of such model is the Ising model, with two values of spin ± 1, it was considered in [6], [8],[17],[18].

The p-adic numbers were introduced by K. Hensel. Many applications of these numbers in theoretical physics have been proposed in papers (see for example, [1],[7],[9],[16],[22]). A number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms [15]. p-adic probability models were investigated in [10],[11]. This is a non-Kolmogorovean model in which probabilities take values in the field of p-adic numbers. This model appears to provide the probabilistic interpretation of p-adic valued wave functions and string amplitudes in the framework of p-adic theoretical physics (see [9],[23]).

In [12],[13] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values has been developed. The non-Archimedean analogue of the Kolmogorov theorem was proved,

that gives the opportunity to construct wide classes of stochastic processes by using finite dimensional probability distributions.

Since the probability theory and stochastic processes in a non-Archimedean setting has been introduced, it is natural to begin the study and the development of the problems of statistical mechanics in the context of the p-adic theory of probability.

One of the central problems in the theory of Gibbs measures is to describe infinite-volume Gibbs measures corresponding to a given Hamiltonian. However, complete analysis of the set of Gibbs measures for a specific Hamiltonian is often a difficult problem. Note, that if for a given Hamiltonian there exist at least two Gibbs measures then a *phase transition* is said to occur for this model.

In this paper we will develop the p-adic probability theory approaches to study inhomogeneous Potts models on a Cayley tree over the field of p-adic numbers. Note in [5] the homogeneous p-adic Potts model with q spin variables on the set of integers $\mathbb{Z}$ has been considered. The aim of this paper is to investigate Gibbs measures and a phase transition problem for the model under consideration.

2 The formal background

2.1 p-adic numbers

Let $\mathbb{Q}$ be the field of rational numbers. Throughout the paper p will be a fixed prime number. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It satisfies the following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,

2) $|xy|_p = |x|_p |y|_p$,

3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

this is a non-Archimedean .

The completion of $\mathbb{Q}$ with respect to the p-adic norm would be a field and it is called *p-adic field* which is denoted by $\mathbb{Q}_p$.

The well-known Ostrovsky's theorem asserts that norms $|x|_\infty = |x|$ and $|x|_p$, $p = 2, 3, 5...$ exhaust all nonequivalent norms on $\mathbb{Q}$ (see [14]). Any p-adic number $x \neq 0$ can be uniquely represented in the canonical series:

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0,1,2,...$ (see more detail [14],[22]). In this case $|x|_p = p^{-\gamma(x)}$.

Let $B(a,r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$, where $a \in \mathbb{Q}_p$, $r > 0$. The p-adic logarithm is defined by series

$$\log_p(x) = \log_p(1+(x-1)) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n},$$

which converges for every $x \in B(1,1)$. And p-adic exponential is defined by

$$\exp_p(x) = \sum_{n=1}^{\infty}\frac{x^n}{n!},$$

which converges for every $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.1. [14],[22] *Let $x \in B(0, p^{-1/(p-1)})$ then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1+x)|_p = |x|_p < p^{-1/(p-1)}$$

and

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1+x.$$

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, ..., A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$ the equality holds

$$\mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n}\mu(A_j).$$

A p-adic measure is called a probability measure if $\mu(X) = 1$. A p-adic probability measure μ is called bounded if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$.

For more detail information about p-adic measures we refer to [10],[11].

2.2 The Cayley tree

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertexes of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *neighboring vertices's*, and we write $l =< x, y >$. A collection of the pairs $< x, x_1 >, ..., < x_{d-1}, y >$ is called *path* from the point x to the point y. The distance $d(x,y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

We set

$$W_n = \{x \in V | d(x, x^0) = n\},$$

$$V_n = \cup_{m=1}^{n} W_m = \{x \in V | d(x, x^0) \leq n\},$$

$$L_n = \{l =< x, y >\in L | x, y \in V_n\},$$

for an arbitrary point $x^0 \in V$. Denote $|x| = d(x, x_0)$, $x \in V$.

Denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\} \quad x \in W_n,$$

this set is called *direct successors* of x. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

Proposition 2.2. [4] *There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$.*

Let us define a group structure on the group Γ_k as follows. Vertices which correspond to the "words" $g, h \in G_k$ are called nearest neighbors and are connected by an edge if either $g = ha_i$ or $h = ga_j$ for some i or j. The graph thus defined is a Cayley tree of order k.

Consider a left (resp. right) transformation shift on G_k defined as: for $g_0 \in G_k$ we set

$$T_{g_0}h = g_0h \ \ (\text{resp.}\ \ T_{g_0}h = hg_0,) \ \ \forall h \in G_k.$$

It is easy to see that the set of all left (resp. right) shifts on G_k is isomorphic to the group G_k.

2.3 The inhomogeneous p-adic Potts model

Let $\mathbb{Q}_p$ be the field of p-adic numbers. By $\mathbb{Q}_p^{q-1}$ we denote $\underbrace{\mathbb{Q}_p \times ... \times \mathbb{Q}_p}_{q-1}$. A norm $\|x\|_p$ of an element $x \in \mathbb{Q}_p^{q-1}$ is defined by $\|x\|_p = \max\limits_{1\leq i\leq q-1}\{|x_i|_p\}$, here $x = (x_1, ..., x_{q-1})$. By xy we understand a bilinear form on $\mathbb{Q}_p^{q-1}$ defined by

$$xy = \sum_{i=1}^{q-1} x_i y_i, \quad x = (x_1, \cdots, x_{q-1}), y = (y_1, \cdots, y_{q-1}).$$

Let $\Psi = \{\sigma_1, \sigma_2, ..., \sigma_q\}$, where $\sigma_1, \sigma_2, ..., \sigma_q$ are elements of $\mathbb{Q}_p^{q-1}$ such that $\|\sigma_i\|_p = 1$, $i = \overline{1, q}$ and

$$\sigma_i\sigma_j = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j \end{cases} \ (i, j = \overline{1, q-1}),\ \sigma_q = \sum_{i=1}^{q-1} \sigma_i.$$

Let $h \in \mathbb{Q}_p^{q-1}$, then we have $h = \sum_{i=1}^{q-1} h_i\sigma_i$ and

$$h\sigma_i = \begin{cases} h_i, & \text{for } i = \overline{1, q-1}, \\ \sum_{i=1}^{q-1} h_i, & \text{for } i = q \end{cases} \tag{2.2}$$

We consider p-adic Potts model where spin takes values in the set Ψ and is assigned to the vertices of the tree. Denote $\emptyset_n = \Psi^{V_n}$, it is the configuration space on V_n. The

Hamiltonian $H_n : \O_n \to \mathbb{Q}_p$ of inhomogeneous p-adic Potts model has the form

$$H_n(\sigma) = -\sum_{<x,y>\in L_n} J_{xy}\delta_{\sigma(x),\sigma(y)}, \quad n \in \mathbb{N}, \tag{2.3}$$

here $\sigma = \{\sigma(x) : x \in V_n\} \in \O_n$, δ is the Kronecker symbol and

$$|J_{xy}|_p < p^{-1/(p-1)}, \quad \forall < x, y > . \tag{2.4}$$

We say (2.3) is *homogeneous Potts model* if $J_{xy} = J, \quad \forall < x, y >$.

3 A construction of Gibbs measures

In this subsection we give a construction of a special class of Gibbs measures for p-adic Potts models on the Cayley tree.

To define Gibbs measure we need in the following

Lemma 3.1. *Let $h_x, x \in V$ be a $\mathbb{Q}_p^{q-1}$-valued function such that $\|h_x\|_p \leq p^{-1/(p-1)}$ for all $x \in V$ and $J_{xy} \in B(0, p^{-1/(p-1)})$, $< x, y >\in L_n$. Then the relation*

$$H_n(\sigma) + \sum_{x\in W_n} h_x\sigma(x) \in B(0, p^{-1/(p-1)})$$

is valid for any $n \in \mathbb{N}$.

The proof easily follows from the strong triangle inequality for the norm $\|\cdot\|_p$.

Let $h : x \in V \to h_x \in \mathbb{Q}_p^{q-1}$ be a function of $x \in V$ such that $\|h_x\|_p < p^{-1/(p-1)}$ for all $x \in V$. Given $n = 1, 2, ...$ consider a p-adic probability measure $\mu_h^{(n)}$ on Ψ^{V_n} defined by

$$\mu_h^{(n)}(\sigma_n) = Z_n^{-1} \exp_p\{-H_n(\sigma_n) + \sum_{x\in W_n} h_x\sigma(x)\}, \tag{3.1}$$

Here, as before, $\sigma_n : x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\tilde{\sigma}_n\in\Omega_{V_n}} \exp_p\{-H_n(\tilde{\sigma}_n) + \sum_{x\in W_n} h_x\tilde{\sigma}(x)\}.$$

Note that according to Lemma 3.1 the measures $\mu^{(n)}$ exist and here the condition (2.4) is necessary to define $\exp_p$.

The compatibility condition for $\mu_h^{(n)}(\sigma_n), n \geq 1$ is given by the equality[1]

$$\sum_{\sigma^{(n)}} \mu_h^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu_h^{(n-1)}(\sigma_{n-1}), \tag{3.2}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$ (cp. [2],[3]).

We note that an analogue of the Kolmogorov extension theorem for distributions can be proved for p-adic distributions given by (3.1) (see [13]). If (3.2) holds for some function

[1] The compatibility condition gives us a possibility to construct a measure μ on whole Ψ^V by means of the measures $\mu^{(n)}$ defined on Ψ^{V_n} such that the restriction of the measure μ to Ψ^{V_n} coincides with the measure $\mu^{(n)}$.

$h = \{h_x : x \in V\}$ then according to the Kolmogorov theorem there exists a unique p-adic measure μ_h depending on h and defined on $\emptyset = \Psi^V$ such that for every $n = 1, 2, ...$ and $\sigma_n \in \Psi^{V_n}$ the equality holds

$$\mu_h\Big(\{\sigma|_{V_n} = \sigma_n\}\Big) = \mu_h^{(n)}(\sigma_n).$$

This μ_h measure is said to be *p-adic Gibbs measure* for the considered Potts model. By $\mathcal{S}$ we denote the set of all p-adic Gibbs measures associated with functions $h = \{h_x,\ x \in V\}$. If $|\mathcal{S}| \geq 2$, then we say that for this model there exists *a phase transition*, otherwise, we say there is *no phase transition* (here $|A|$ means the cardinality of a set A). Now our problem is to find for what kind of functions $h = \{h_x : x \in V\}$ the measures defined by (3.1) would satisfy the compatibility condition (3.2). Of course, there are many functions h_x for which the condition (3.2) is not satisfied. For example, let the order of the tree be 2, i.e. $\Gamma^2 = \mathbb{Z}$, and consider homogeneous p-adic Potts model with $q = 2$. In this case the model reduces to the well-known Ising model with spin values ± 1. Now for an arbitrary non-zero $h \in \mathbb{Q}_p$, $|h|_p \leq p^{-1/(p-1)}$ set

$$h_n = \begin{cases} h, & \text{if} \quad n = 2k, \\ 0, & \text{if} \quad n = 2k+1. \end{cases}$$

Then it is not hard to verify that the corresponding measures (see (3.1)) associated with the function $h = \{h_n,\ n \in \mathbb{Z}\}$ do not satisfy the condition (3.2).

The following statement describes conditions on h_x guaranteeing the compatibility condition for the measures $\mu_h^{(n)}(\sigma_n)$.

Theorem 3.2. *The measures $\mu_h^{(n)}(\sigma_n)$, $n = 1, 2, ...$ satisfy the compatibility condition (3.2) if and only if for any $x \in V$ the following equation holds:*

$$h'_x = \sum_{y \in S(x)} F(h'_y; \theta_{xy}, q) \tag{3.3}$$

here and below $\theta_{xy} = \exp_p(J_{xy})$, a vector $h' = (h'_1, ..., h'_{q-1})$ is defined by a vector $h = (h_1, ..., h_{q-1})$ as follows $h'_i = \sum\limits_{j=1, j\neq i}^{q-1} h_j$, $i = 1, ..., q-1$ and a mapping $F : \mathbb{Q}_p^{q-1} \to \mathbb{Q}_p^{q-1}$ is $F(h; \theta, q) = (F_1(h; \theta, q), ..., F_{q-1}(h; \theta, q))$ with

$$F_i(h; \theta, q) = F_i(h_1, ..., h_{q-1}; \theta, q) = \log_p\Big[\frac{(\theta - 1)\exp_p(h_i) + \sum_{j=1}^{q-1}\exp_p(h_j) + 1}{\sum_{j=1}^{q-1}\exp_p(h_j) + \theta}\Big],$$

where $i = 1, ..., q-1$, and $\theta \in \mathbb{Q}_p$.

Proof. Using (2.2) it is easy to see that (3.2) and (3.3) are equivalent.(cf. [18],[19]).

Observe that according to this Theorem the problem of describing of p-adic Gibbs measures reduces to the describing of solutions of functional equation (3.3).

4 Uniqueness of the Gibbs measure

In this section we will prove the following

Theorem 4.1. *If $q \notin p\mathbf{N}$ then the equation (3.3) has unique solution $h_x = (0, ..., 0) \in \mathbb{Q}_p^{q-1}$, $\forall x \in V$, for every $k \geq 1$ and J_{xy} with (2.4), i.e. $|\mathcal{S}| = 1$.*

In order to prove this Theorem we will prove some auxiliary lemmas.

The following lemma plays the key role in our analysis.

Lemma 4.2. *If $|a_i - 1|_p \leq M$ and $|a_i|_p = 1$, $i = 1, ..., n$, then*

$$\Big|\prod_{i=1}^{n} a_i - 1\Big|_p \leq M. \tag{4.1}$$

Proof. We prove by induction on n. The case $n = 1$ is the condition of lemma. Suppose that (4.1) is valid at $n = m$. Now let $n = m + 1$. Then we have

$$\Big|\prod_{i=1}^{m+1} a_i - 1\Big|_p = \Big|\prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m} a_i + \prod_{i=1}^{m} a_i - 1\Big|_p \leq$$

$$\leq \max\left\{\Big|\prod_{i=1}^{n} a_i(a_{n+1} - 1)\Big|_p, \Big|\prod_{i=1}^{n} a_i - 1\Big|_p\right\} \leq M$$

This completes the proof.

Let $h_x = (h_{1,x}, \cdots, h_{q-1,x})$, $x \in V$ be a solution of (3.3). Denote $z_x = (z_{1,x}, ..., z_{q-1,x})$, where $z_{i,x} = \exp_p(h'_{i,x})$ or $z_{i,x} = \exp_p\Big(\sum\limits_{j=1, j\neq i}^{q-1} h_{j,x}\Big)$, $i = 1, \cdots, q-1$. Then the equation (3.3) can be written as follows

$$z_{i,x} = \prod_{y \in S(x)} \frac{(\theta_{xy} - 1)z_{i,y} + \sum_{j=1}^{q-1} z_{j,y} + 1}{\sum_{j=1}^{q-1} z_{j,y} + \theta_{xy}}, \quad i = 1, ..., q-1 \tag{4.2}$$

Let $S(x) = \{x_1, ..., x_k\}$, here as before $S(x)$ is the set of direct successors of x. Using this notation we rewrite the equation (4.2) as

$$z_{i,x} = \prod_{m=1}^{k} a_{i,x,m},$$

where $a_{i,x,m} = \dfrac{(\theta_{xx_m} - 1)z_{i,x_m} + \sum_{j=1}^{q-1} z_{j,x_m} + 1}{\sum_{j=1}^{q-1} z_{j,x_m} + \theta_{xx_m}}$, $\theta_{xx_m} = \exp_p(J_{xx_m})$, $m = 1, ..., k$, $x \in V$.

Lemma 4.3. *For every $x \in V$ the following inequality holds*

$$|h'_{i,x}|_p \leq \frac{1}{p} \max_{1 \leq j \leq k} \{|h'_{i,x_j}|_p\}.$$

Proof. For every $m \in \{1, 2, ..., k\}$ and $i = 1, ..., q-1$ we have

$$|a_{i,x,m} - 1|_p = \Big|\frac{(\theta_{xx_m} - 1)(z_{i,x_m} - 1)}{\sum_{j=1}^{q-1} z_{j,x_m} + \theta_{xx_m}}\Big|_p =$$

$$= \Big|\frac{(\theta_{xx_m} - 1)(z_{i,x_m} - 1)}{\sum_{j=1}^{q-1}(z_{j,x_m} - 1) + (\theta_{xx_m} - 1) + q}\Big|_p \leq \frac{1}{p}|h'_{i,x_m}|_p.$$

Also

$$|a_{i,x,m}|_p = \Big|\frac{(\theta_{xx_m} - 1)z_{i,x_m} + \sum_{j=1}^{q-1}(z_{j,x_m} - 1) + q}{\sum_{j=1}^{q-1}(z_{j,x_m} - 1) + (\theta_{xx_m} - 1) + q}\Big|_p = 1.$$

Here and above we have used the equality $|\exp_p(x) - 1|_p = |x|_p$ (see Lemma 2.1), the condition (2.4) and $|q|_p = 1$.

Hence the conditions of Lemma 4.2 are satisfied for $a_{i,x,m}, m = 1, ..., k$, whence

$$|h'_{i,x}|_p = |z_{i,x} - 1|_p = \Big|\prod_{m=1}^{k} a_{i,x,m} - 1\Big| \leq \frac{1}{p} \max_{1\leq j\leq k}\{|h'_{i,x_j}|_p\},$$

this completes the proof.

Lemma 4.4. *If $h'_x = 0$ then $h_x = 0$.*

The proof follows from the equality

$$h_{k,x} = \frac{1}{q-2}\sum_{i=1}^{q-1} h'_{i,x} - h'_{k,x}, \tag{4.3}$$

where $k = 1, \cdots, q-1$.

Proof of Theorem 4.1. It is easy to see that $h_x = (0, ..., 0), x \in V$ is a solution of (3.3). We want to prove that any other solution of (3.3) coincides with this one.

Now let $h_x = (h_{1,x}, \cdots .h_{q-1,x}), x \in V$ be a solution of (3.3). Take an arbitrary $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{p^{n_0}} < \varepsilon$. According to Lemma 4.3 we have

$$\|h'_x\|_p \leq \frac{1}{p}\|h'_{x_{i_0}}\|_p \leq \frac{1}{p^2}\|h_{x_{i_0,i_1}}\|_p \leq \cdots$$
$$\leq \frac{1}{p^{n_0-1}}\|h_{x_{i_0,\dots,i_{n_0-2}}}\|_p \leq \frac{1}{p^{n_0}} < \varepsilon,$$

here $x_{i_0,...,i_n,j}$, $j = 1, .., k$ are direct successors of $x_{i_0,...,i_n}$ and

$$\|h_{x_{i_0,\dots,i_m}}\|_p = \max_{1\leq j\leq k}\{\|h_{x_{i_0,\dots,i_{m-1},j}}\|_p\}.$$

The arbitrariness of ε implies that $h'_x = 0$, hence Lemma 4.4. yields $h_x = 0$ for every $x \in V$. The theorem is proved.

5 Non-uniqueness of the Gibbs measure

In this section, we shall prove non-uniqueness of Gibbs measure for $q \in p\mathbb{N}$. Denote

$$\Lambda = \{h = (h_x \in \mathbb{Q}_p^{q-1}, x \in V) : h_x \text{ satisfies the equation (3.3)}\}.$$

To prove that the Gibbs measure is not unique it is enough to show that set Λ contains at least two distinct elements. The description of an arbitrary elements of the set Λ is a complicated problem.

In this section, we restrict ourselves to description of periodic elements of Λ.

Let G_k be a free product of $k+1$ cyclic groups of order two. According to Proposition 2.2 there is a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k. Let $\hat{G}_k \subset G_k$ be a normal subgroup of finite index.

We say that $h = \{h_x : x \in G_k\}$ is *$\hat{G}_k$-periodic* if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in \hat{G}_k$. A Gibbs measure is called *$\hat{G}_k$-periodic* if it corresponds to $\hat{G}_k$-periodic function h. Note that G_k-periodic Gibbs measures are called *translation-invariant*.

Let H_0 be a subgroup of index r in G_k, and let $G_k|H_0 = \{H_0, H_1, ..., H_{r-1}\}$ be the quotient group. Let $q_i(x) = |S^*(x) \cap H_i|, i = 0, 1, ..., r-1$; $N(x) = |\{j : q_j(x) \neq 0\}|$, where $S^*(x)$ is the set of all nearest neighbors of $x \in G_k$. Denote $Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x))$. We note (see [20]) that for every $x \in G_2$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e is the identity of G_k) such that

$$\pi_x Q(e) = Q(x).$$

It follows from this equality that $N(x) = N(e)$ for all $x \in G_k$.

Each H_0- periodic function is given by

$$\{h_x = h^{(i)} \text{ for } x \in H_i, i = 0, 1, ..., r-1\}.$$

Let G_k^* be the subgroup in G_k consisting of all words of even length. Clearly, G_k^* is a subgroup of index 2. We are interested in the answer to the following question: whether there exist at least two G_k^*-periodic p-adic Gibbs measures?

For the simplicity we will consider the cases $k = 1, 2$ with $q \in p\mathbb{N}$.

`Case:` $k = 1$. In this case, we assume

$$\theta_{xy} = \begin{cases} \theta_1 & \text{if } x \in G_1^*, \quad y \in G_1 \setminus G_1^*, \\ \theta_2 & \text{if } x \in G_1 \setminus G_1^*, \quad y \in G_1^* \end{cases} \tag{5.1}$$

and

$$h_x = \begin{cases} h = (h_1, ..., h_{q-1}) & \text{if } x \in G_1^*, \\ l = (l_1, ..., l_{q-1}) & \text{if } x \in G_1 \setminus G_1^*, \end{cases}$$

Then from (3.3) we have

$$\begin{cases} z_i = \dfrac{(\theta_1-1)t_i+\sum_{j=1}^{q-1}t_j+1}{\sum_{j=1}^{q-1}t_j+\theta_1} \\ t_i = \dfrac{(\theta_2-1)z_i+\sum_{j=1}^{q-1}z_j+1}{\sum_{j=1}^{q-1}z_j+\theta_2} \end{cases} \tag{5.2}$$

where $z_i=\exp_p(h'_i)$ and $t_i=\exp_p(l'_i)$. Observe that if $z_i=1$ then $t_i=1$ (the converse is also true). Denote $\alpha=\frac{\theta_1\theta_2+q-1}{\theta_1+\theta_2+q-2}$, then substituting $t_i=1,\quad i=2,...,q-1$ into (5.2) we get

$$z=\frac{\alpha z+q-1}{z+\alpha+q-2}, \tag{5.3}$$

with $z=z_1$.

It is easy to see that the equation (5.3) has two solutions $z=1$ and $z=1-q$ for any $\alpha\neq 1$ and $z=1$ if $\alpha=1$.

Note that a solution of (5.3) will define a p-adic Gibbs measure if it satisfies the inequality $|z-1|_p\leq\frac{1}{p}$, which is true for $z=1$ and $z=1-q$, since $q\in p\mathbb{N}$. Thus the system of equations (5.2) has two solutions $z_i=1$, $t_i=1$, $i=1,\cdots,q-1$ and $z_1=1-q, z_j=1$, $t_1=1-q, t_j=1$, $j=2,\cdots,q-1$ if $\alpha\neq 1$. Consequently, the equation (3.3) has two solutions (see (4.3)).

Remark 5.1. It is known [21] that for the Potts model and even for arbitrary models on $\mathbb{Z}$ with finite radius of interaction of the particles, in which $\mathbb{R}$ is considered instead of $\mathbb{Q}_p$, there are no phase transitions. In the case under consideration this pattern is destroyed.

`Case:` $k=2$. In this case for simplicity we will assume that $\theta_{xy}=\theta$ for any $<x,y>$. Then the problem of describing G_2^*-periodic Gibbs measures reduces to the description of solutions of the following equation:

$$\begin{cases} h^{(1)'}=F(h^{(2)'};\theta,q) \\ h^{(2)'}=F(h^{(1)'};\theta,q), \end{cases} \tag{5.4}$$

where $h^{(1)'},h^{(2)'}\in\mathbb{Q}_p^{q-1}$ and the map F is defined in Theorem 3.2.

Observe that for every $l=1,...,q-1$ $h_l^{(i)'}=0$ satisfies l-th equation, where $h^{(i)'}=(h_1^{(i)'},\cdots,h_{q-1}^{(i)'})$, $i=1,2$. Denoting $z_i=\exp_p(h_1^{(i)'})$ and substituting $h_l^{(i)'}=0$ at $l=2,...,q-1$ to the first equation of (5.4) we obtain

$$\begin{cases} z_1=f(z_2) \\ z_2=f(z_1), \end{cases} \tag{5.5}$$

here

$$f(z)=\Big(\frac{\theta z+q-1}{z+\theta+q-2}\Big)^2. \tag{5.6}$$

The first equation in (5.5) can be written as follows:

$$(\theta^2 - z_1)z_2^2 + [2\theta(q-1) - 2(\theta+q-2)z_1]z_2 - (\theta+q-2)^2 z_1 + (q-1)^2 = 0. \tag{5.7}$$

We know that if $z_2 = z_1$ then (5.7) reduces to the equation

$$z_1^3 + (2q - (\theta-1)^2 - 3)z_1^2 + ((\theta-1)^2 + 3 + q^2 - 4q)z_1 - (q-1)^2 = 0, \tag{5.8}$$

which describes the translation-invariant Gibbs measures.

We add and subtract (5.8) in (5.7), consequently, we have

$$(z_2 - z_1)\{(\theta^2 - z_1)z_2 - z_1^2 + (\theta^2 - 2\theta + 4 - 2q)z_1 + 2\theta(q-1)\} -$$
$$-[z_1^3 + (2q - (\theta-1)^2 - 3)z_1^2 + ((\theta-1)^2 + 3 + q^2 - 4q)z_1 - (q-1)^2] = 0. \tag{5.9}$$

Now substituting $z_2 = f(z_1)$ in (5.9) and noticing that the numerator of $z_2 - z_1 = f(z_1) - z_1$ equal to the sentence in the square brackets, so cutting them out we get

$$(\theta^2 - z_1)f(z_1) - z_1^2 + (\theta^2 - 2\theta + 4 - 2q)z_1 + 2\theta(q-1) + (z_1 + \theta + q - 2)^2 = 0.$$

or

$$(-\theta^2 + z_1)f(z_1) = \theta^2 z_1 + (\theta^2 + q^2 + 4\theta q - 6\theta - 4q + 4).$$

Hence we get

$$(\theta^2 + \theta + q - 2)^2 z_1^2 +$$
$$[\theta^4 + 4(q-1)\theta^3 + (q^2 + 6q - 12)\theta^2 + 2(5q^2 - 18q + 16)\theta + (2q^3 - 13q^2 + 26q - 17)]z_1 +$$
$$+[\theta(q-1) + (\theta + q - 2)^2]^2 = 0. \tag{5.10}$$

Denote

$$\alpha = (\theta^2 + \theta + q - 2)^2,$$
$$\beta = \theta^4 + 4(q-1)\theta^3 + (q^2 + 6q - 12)\theta^2 + 2(5q^2 - 18q + 16)\theta + (2q^3 - 13q^2 + 26q - 17),$$
$$\tau = [\theta(q-1) + (\theta + q - 2)^2]^2.$$

The following equalities

$$\alpha = [(\theta^2 - 1) + (\theta - 1) + q]^2,$$
$$\beta = (\theta^4 - 1) + 4(q-1)(\theta^3 - 1) + (q^2 + 6q - 12)(\theta^2 - 1) + 2(5q^2 - 18q + 16)(\theta - 1) + 2q^3 - 2q^2,$$
$$\tau = [3(q-1)(\theta-1) + (\theta-1)^2 + (q-1)^2]^2$$

and the inequality $|\theta^i - 1|_p \leq \frac{1}{p}$, $i \in \mathbb{N}$ imply with the strong triangle property that

$$|\alpha|_p \leq \frac{1}{p}, \quad |\beta|_p \leq \frac{1}{p}, \quad |\tau|_p = 1, \tag{5.11}$$

here we have used that $q \in p\mathbb{N}$.

A solution z of (5.10) will define a p-adic Gibbs measure if it satisfies the inequality $|z - 1|_p \leq \frac{1}{p}$. So, we should check the last condition. Rewrite the equation (5.10) as follows

$$\alpha(z_1^2 - 1) + \beta(z_1 - 1) + \alpha + \beta + \tau = 0. \tag{5.12}$$

The equalities (5.11) imply that

$$|\alpha(z_1^2-1)+\beta(z_1-1)|\le\frac{1}{p},\quad |\alpha+\beta+\tau|_p=1$$

this means, that the equation (5.12) has no solutions, which satisfy $|z-1|_p\le\frac{1}{p}$.

Now, for the simplicity we restrict ourselves to the case: $q=3$ and $p=3$. In this case, it is easy to see that the equation (5.4) can have solutions only of the form $(h,0)$, $(0,h)$ and (h,h). According to above made argument, one can see that the equation (5.4) has no solution like $(h,0)$ and $(0,h)$. Solutions of the form (h,h) describes only translation-invariant Gibbs measures.

Summarizing, we obtain the following

Theorem 5.1. *(i) For $k=1$ and $q\in pN$ any G_1^*- periodic Gibbs measures of inhomogeneous p-adic Potts model with condition (5.1) coincide with translation-invariant Gibbs measures. If $(\theta_1-1)(\theta_2-1)\ne 0$ then there occurs a phase transition.*

(ii) If $k=2$, $p=3$, $q=3$ then for 3-adic homogeneous Potts model G_2^*-periodic Gibbs measures coincide with translation-invariant ones which correspond to the solutions of the equation (5.8), in this case there is a phase transition.

Remark 5.2. In the real case it is known [18] that the G_2^*-periodic Gibbs measures are different from translation-invariant ones even for the homogeneous Ising model on the Cayley tree Γ^2.

Note, that existence of translation-invariant Gibbs measures for homogeneous p-adic Potts models has been proved in [5] for $k=1$ and in [19] for $k=2$. The main results of these papers are the following

Theorem 5.2. [5] *If $k=1$ and $q\in p\mathbb{N}$, $p>2$ then for p-adic Potts model there are (phase transitions) at least two translation-invariant Gibbs measures.*

Theorem 5.3. [19] (i) *Let $p=2$, $q\in 2^2\mathbb{N}$ and $J\ne 0$. If $q=2^2s$, $(s,2)=1$, $|J|_2=\dfrac{1}{4}$ or $q=2^m s$, $m\ge 3$, $(s,2)=1$, $|J|_2\le\dfrac{1}{4}$ then there exists (phase transition) at least two translation-invariant Gibbs measures for the homogeneous 2-adic Potts model on a Cayley tree of order 2.*

(ii) *Let $p\ge 3$, $q\in p\mathbb{N}$, and $0<|J|_p\le\dfrac{1}{p}$ then there exist at least q translation-invariant Gibbs measures for the homogeneous p-adic Potts model on a Cayley tree of order 2.*

(iii) *The p-adic Gibbs measure corresponding to the homogeneous p-adic Potts model on the Cayley tree of order k is bounded if and only if $q\notin p\mathbb{N}$.*

Conjecture. For any $k\ge 1$, $q\in p\mathbb{N}$ and any subgroup of finite index, each periodic Gibbs measure of the homogeneous p-adic Potts model is translation-invariant.

Remark 5.3. If $q \notin p\mathbb{N}$ then Theorem 5.3 says that the p-adic Gibbs measure corresponding to the Potts model is bounded, hence in this case there is (unique) bounded p-adic Gibbs measure for the p-adic Potts model.

Remark 5.4. If $q = 2$ then p-adic Potts model becomes the p-adic Ising model. Hence, Theorems 4.1 and 5.3 imply for $p \geq 3$ that for the inhomogeneous p-adic Ising model there is unique Gibbs measure which is translation-invariant and bounded. The description of Gibbs measures for the inhomogeneous p-adic Ising model with $p = 2$ would be given elsewhere.

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